Double Phase Slips and Bound Defect Pairs in Parametrically Driven Waves*

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Spatio-temporal chaos in parametrically driven waves is investigated in one and two dimensions using numerical simulations of Ginzburg-Landau equations. A regime is identified in which in one dimension the dynamics are due to double phase slips. In very small systems they are found to arise through a Hopf bifurcation off a mixed mode. In large systems they can lead to a state of localized spatio-temporal chaos, which can be understood within the framework of phase dynamics. In two dimensions the double phase slips are replaced by bound defect pairs. Our simulations indicate the possibility of an unbinding transition of these pairs, which is associated with a transition from ordered to disordered defect chaos.

I. INTRODUCTION

While low-dimensional chaotic dynamics are quite well understood this is not the case for chaotic dynamics of high dimension as, for instance, spatio-temporal chaos. Spatio-temporal chaos arises in systems in which spatial degrees of freedom play an important role and the structures are not only chaotic in time but also in space. These dynamics arise often in pattern-forming systems when all ordered patterns become unstable. Of particular interest are systems in which the chaos is extensive, i.e. systems in which quantities like the attractor dimension and the number of positive Lyapunov exponents grow linearly with the system size, so that one can think of them as a large number of coupled, chaotic entities.

Spatio-temporal chaos is found in many experimental systems. It has been extensively studied in convection where a number of different types have been observed. In the presence of rotation, a classic result is the occurrence of domain chaos. It is due to the Küppers-Lortz instability which renders steady convection rolls unstable to rolls with a different orientation [1]. Since the new rolls are susceptible to the same instability a persistent switching of patches of rolls of different orientations is observed [2]. Very recently spatio-temporal chaos has also been found (without rotation) in regimes in which the convection rolls are in fact linearly stable. Sufficiently large perturbations, however, lead to a state of spiral-defect chaos in which spirals and other types of defects dominate [3].

Another type of spatio-temporal chaos arises in electroconvection in nematic liquid crystals in a traveling wave regime [4]. Due to the axial anisotropy of this system the waves travel only in certain directions relative to the axis of anisotropy. In the regime in question they travel obliquely to that direction and because of the reflection symmetries of the system the dynamics is governed by the competition of waves traveling in 4 directions. The chaotic dynamics arise immediately at the onset of convection and is characterized by defects in the various wave components. Associated with each defect is a suppression of the corresponding wave amplitude leading to domains in which one or two of the wave components dominate. Some understanding of these dynamics and their possible origin has been obtained within a set of coupled Ginzburg-Landau equations [5]. Spatio-temporal chaos of traveling waves in an isotropic system arise in convection in binary mixtures [6].

A very rich class of pattern-forming systems are parametrically driven waves, the standard realization being surface waves on a fluid that are excited by a vertical shaking of the container at twice the frequency of the waves. Depending on the fluid parameters and the driving frequency spatially periodic and quasi-periodic

*To appear in the Proceedings of the *15th Symposium on Energy Engineering Sciences*, Office of Basic Energy Sciences, Department of Energy, 1997.

patterns of various kinds have been found as well as transitions to spatio-temporal chaos [7]. Strikingly, very similar phenomena can be found even if the fluid is replaced by a granular medium like sand [8].

In the present communication we present theoretical results for the dynamics of parametrically driven waves in one and two dimensions within the framework of Ginzburg-Landau equations. In the one-dimensional analysis we address the important question how to characterize the behavior of a spatially and temporally chaotic state on length scales that are much larger than the typical wavelength. The response of regular periodic patterns to long-wave perturbations is well understood; in some sense it is a dissipative analogue to sound waves in crystals. For steady patterns the response is typically diffusive whereas for oscillatory patterns it is propagative. Both can be described by phase equations (or equations for the local wavenumber). Here we describe a chaotic state for which the same type of description is possible, i.e. although the dynamics on short scales is chaotic in space and time, the large-scale behavior of that state is diffusive. This striking behavior is due to the fact that the chaotic state is driven by *double phase slips* as described below. We show that the homogeneously chaotic state can become diffusively unstable on large scales and separates into arrays of chaotic and non-chaotic domains very similar to phase separation in mixtures. This provides a mechanism that can lead to the *localization* of the chaotic dynamics in space.

Experimentally, localized spatio-temporal chaos has been observed in Taylor vortex flow [9], Rayleigh-Bénard convection [10], and parametrically excited surface waves [11]. So far, the localization mechanism in these systems is, however, only poorly understood.

In the second part we present ongoing work on the dynamics that arise in two dimensions in the regime in which double-phase slips occur in one dimension. As discussed below it leads to 'fluctuating bound defect pairs' . We present evidence that indicates a transition from an ordered state of defect chaos to a disordered one. This transition appears to be associated with an 'unbinding' of the defect pairs.

II. GINZBURG-LANDAU EQUATIONS FOR PARAMETRICALLY DRIVEN WAVES

To obtain a tractable model for parametrically driven waves we consider systems that exhibit a supercritical Hopf bifurcation to traveling waves, i.e. when the control parameter is increased beyond a certain threshold value the basic state becomes unstable to small-amplitude traveling waves. This is, for instance, the case in electroconvection of nematic liquid crystals [4][1]. Just below the Hopf bifurcation the traveling-wave modes are only weakly damped. Therefore a small driving is sufficient to excite standing waves of small amplitude. Consequently, a weakly nonlinear description is possible by expanding about the unforced basic state and treating the forcing and the damping as small perturbations. This leads to two coupled equations for the complex amplitudes of the traveling-wave components, i.e. physical quantities like the vertical fluid velocity u in the midplane of a convection system are given by

$$u(\mathbf{r},t) = \epsilon A(X,Y,T)e^{i(\mathbf{q}_c\cdot\mathbf{r}-\frac{\omega_c}{2}t)} + \epsilon B(X,Y,T)e^{i(\mathbf{q}_c\cdot\mathbf{r}+\frac{\omega_c}{2}t)} + c.c. + o(\epsilon). \tag{1}$$

The complex amplitudes A and B vary on the slow time and space scales, $T = \epsilon^2 t$, $X = \epsilon x$, and $Y = \epsilon y$ with $\mathbf{r} = (x, y)$, and $\epsilon \ll 1$.

Using standard symmetry arguments one obtains for the amplitudes A and B in one dimension the Ginzburg-Landau equations [13]

$$\partial_T A + s\partial_X A = d\partial_X^2 A + aA + bB + cA(|A|^2 + |B|^2) + gA|B|^2, \tag{2}$$
$$\partial_T B - s\partial_X B = d^*\partial_X^2 B + a^*B + bA + c^*B(|A|^2 + |B|^2) + g^*B|A|^2. \tag{3}$$

[1] Convection in binary mixtures also exhibits a Hopf bifurcation to traveling waves. It is, however, subcritical [6] and the waves appear right away with finite amplitude. An analysis of the parametric forcing of that system is therefore more complicated [12].

The coefficients in (2,3) are complex except for s and b, which are real. The real part a_r of the coefficient of the linear term a gives the linear damping of the traveling waves in the absence of the periodic forcing and is proportional to the distance from the Hopf bifurcation. The coefficient of the linear coupling term b gives the amplitude of the periodic forcing as can be seen from the fact that it breaks the continuous time-translation symmetry $t \rightarrow t + \Delta t$ that implies the transformation $A \rightarrow Ae^{i\Delta t\omega_e/2}$ $B \rightarrow Be^{-i\Delta t\omega_e/2}$. The imaginary part a_i of the coefficient of the linear term a gives the difference between the frequency of the unforced waves and half the forcing frequency ω_e.

The same Ginzburg-Landau equations are obtained also for systems that do not exhibit a Hopf bifurcation, if they have weakly damped traveling wave modes. This is the case for surface waves on fluids with small viscosity. Then again only a small driving is necessary to excite the surface waves and one can perform the same kind of expansions. In that case $a_r < 0$ represents the damping of the waves and the group velocity parameter s is in general complex, indicating that the dissipation depends already linearly on the wavenumber. However, all dissipative terms are small.

In addition to the trivial solution $A = B = 0$ (2,3) possess three types of simple solutions: $|A| = |B| = const$, $|A| \neq |B|$ (both constant), and $|A| = |B|$ (both time-periodic). We are in particular interested in the first type, which corresponds in the physical system to standing waves which are phase-locked to the parametric forcing, i.e. they are excited by the forcing. With increasing a_r they become unstable to solutions of the second type, which correspond to traveling waves as they exist also in the absence of the periodic forcing. Solutions of the third type correspond to standing waves that are not phase-locked to the forcing. For $g_r < 0$ they are unstable to the traveling waves.

The response of the phase-locked standing waves to long-wave perturbations can be described using a phase equation [14],

$$\partial_T \phi = D(q)\partial_X^2 \phi \qquad \text{with } q = \partial_X \phi, \tag{4}$$

which due to the spatial reflection symmetry of the waves is a (nonlinear) diffusion equation. The diffusion coefficient is not necessarily positive and its sign-change indicates an instability of the waves, the Eckhaus instability. For the one-dimensional case the diffusion coefficient was given in [14]. The resulting stability limits as they are relevant for the first part of the paper are shown in fig.1a. The neutral curve, above which the basic state is unstable to standing-wave perturbations, is given by the dashed line. The solid line gives the Eckhaus instability of the phase-locked standing waves. In the second part of the paper we will consider a case in which traveling waves appear in the absence of forcing, i.e. $a_r > 0$. The corresponding stability regions are shown in fig.1b. The neutral curve is given by the dashed line, neutral curve for the appearance of traveling waves by a dotted line. Their Eckhaus instability is denoted by a solid line. In addition to the Eckhaus instability also a parity-breaking instability arises in which the standing waves become unstable to traveling waves. It is indicated by a dashed-dotted line. Over some range of parameters the parity-breaking instability is preempted by a mode that arises first at finite modulation wavenumber (open squares). It emerges from the parity-breaking instability. The standing waves are stable only inside the region marked by the solid lines and the squares.

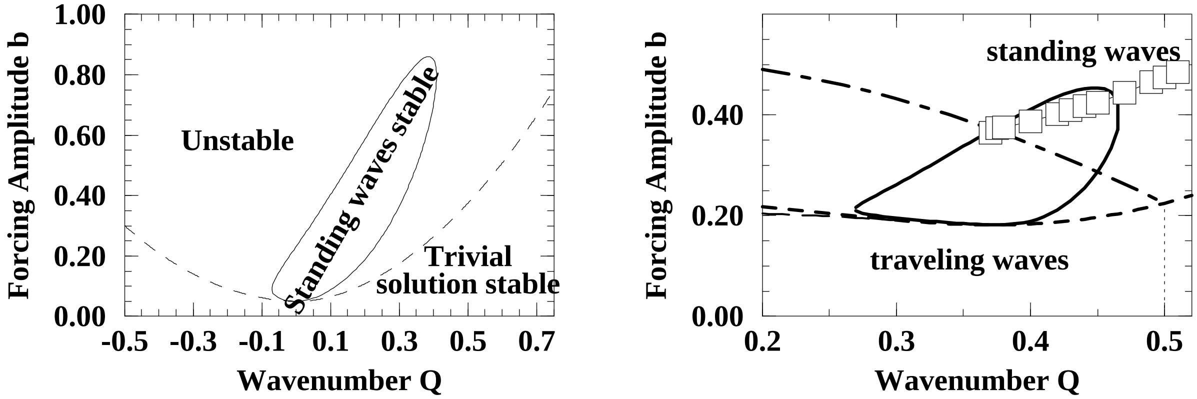
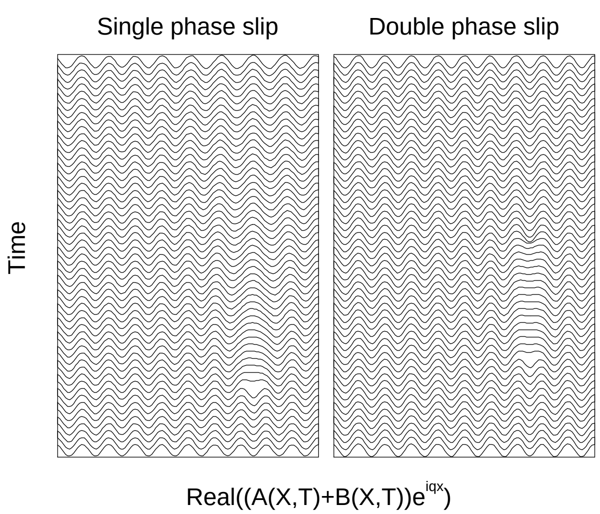


FIG. 1. Linear stability diagram for $c = -1 + 4i$, $d = 1 + 0.5i$, $s = 0.2$, $g = -1 - 12i$. a) $a = -0.05$, b) $a_r = 0.25$.

III. DYNAMICS IN A SMALL SYSTEM

The central new feature of the standing waves to be discussed in this paper is the appearance of 'double phase slips' [15,16]. Usually the Eckhaus instability leads to a single phase slip which changes the total phase in the system and through which the wavenumber of the pattern jumps from outside the stable band to inside the band. Such an event is shown in fig.2a. While this occurs also in this system when the Eckhaus instability is crossed for weak forcing, it is not the case for larger forcing: for $b \approx 0.4$ and above the same perturbation leads to a double phase slip, which consists of two consecutive phase slips that undo each other as shown in fig.2b. After the double phase slip the pattern has the same wave number as before and can undergo the same instability again leading to persistent dynamics. It can be periodic or irregular as discussed in sec.IV.



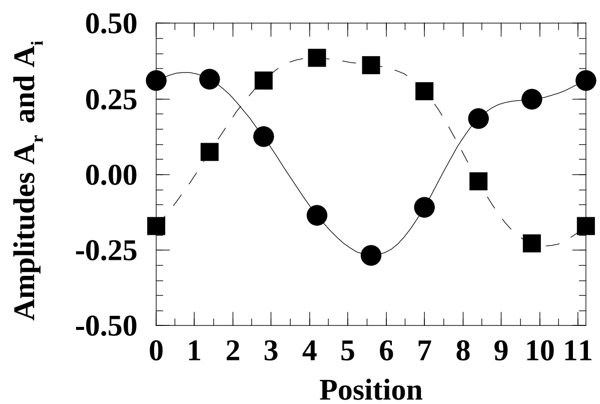


FIG. 2. Space-time diagrams showing a) single phase slip for small values of the forcing amplitude (e.g. $b = 0.1$), and b) double phase slip observed for larger values of the forcing amplitude (e.g. $b = 0.6$). Other parameters as in fig.1a.

FIG. 3. Mixed-mode solution for $L = 11.2$

To study the origin of the double phase slip we consider a minimal system in which only the Fourier modes 0 and 1 are important and simulate (2,3) numerically with a pseudospectral code keeping only the

Fourier modes -4 to +4. Changing the length of the system at fixed $b = 1.0$ we find the phase portraits shown in fig.4a and fig.4b. In each run the same pattern very close to the solution with one wavelength in the system is chosen as initial condition (solid diamond). For small length (large wavenumber) this initial condition rapidly evolves to the homogeneous solution (short dashes and open triangle).

When the length is increased the homogeneous solution becomes unstable and a new stable fixed point corresponding to a mixed mode involving Fourier modes 0 as well as 1 arises (open circle). Consequently the solution converges to that mixed mode. Fig.3 shows the real and imaginary parts of the amplitude A of the mixed mode for $L = 11.2$. With increasing L the mixed-mode fixed point moves to the left and the trajectory turns around. For $L = 11.2$ the mixed-mode fixed point, which was a stable node for smaller L, becomes a stable spiral point (open square). When L is increased to $L = 11.3$ the spiral point becomes unstable and generates a stable limit cycle as is shown in fig.4b. Closer inspection of that phase portrait shows that the trajectory follows a three-dimensional path: while it starts *outside* the limit cycle it intersects itself a number of times and eventually approaches the limit cycle from *inside*. This indicates that a model that is to capture the mixed mode, its limit cycle, and the two fixed points corresponding to the homogeneous solution and that with one wavelength has to be at least three-dimensional. For yet larger L, a second mixed-mode fixed point appears, which is associated with the Eckhaus instability of the solution with one wavelength. It becomes important when the limit cycle grows and eventually becomes homoclinic to this additional fixed point around $L = 12$.

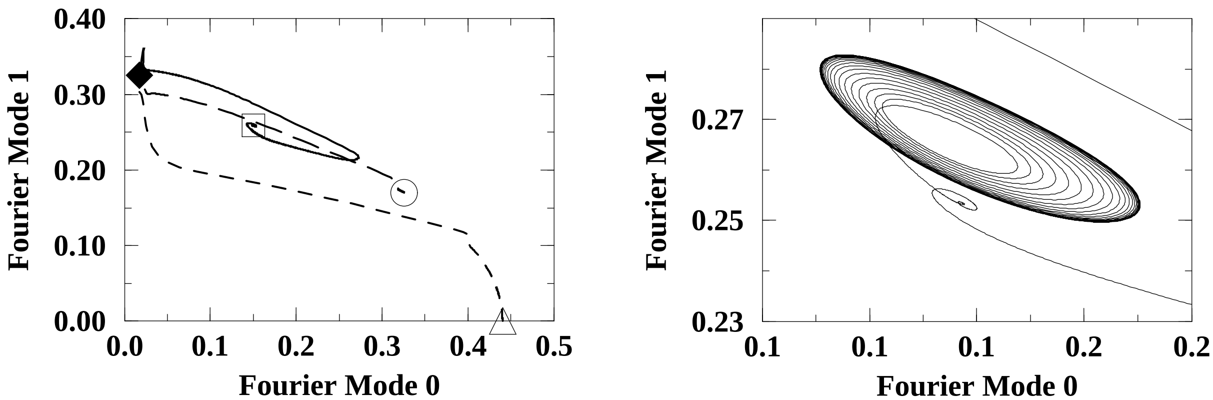


FIG. 4. Phase space projected onto the magnitude of the Fourier modes 0 and 1 for increasing values of the system length L. a) $L = 8$ (short dashes), $L = 10$ (long dashes), and $L = 11.2$ (solid line). Initial condition is marked by a solid diamond, the final fixed point by open symbols. b) $L = 11.3$.

IV. DYNAMICS IN A LARGE SYSTEM

While in the small system discussed in sec.III only simple dynamics was found, complex dynamics arises if the double phase slips can occur at more than one location. Here we describe results for large systems where a surprisingly simple description of the large-scale behavior is possible. Our search for such a description was motivated by states of *localized* spatio-temporal chaos [15,16]. An example is shown in fig.5 where each of the double phase slips is marked as a dot. An initial perturbation triggers a double phase slip. In its vicinity more double phase slips arise and the chaotic activity starts to spread. However, by $t = 50,000$ the width of the chaotic domain stops growing and a stable state is reached, in which the chaotic activity is confined to part of the homogeneous system. At first sight, this result is very surprising; one might have expected that the chaotic activity would always spread through the whole system.

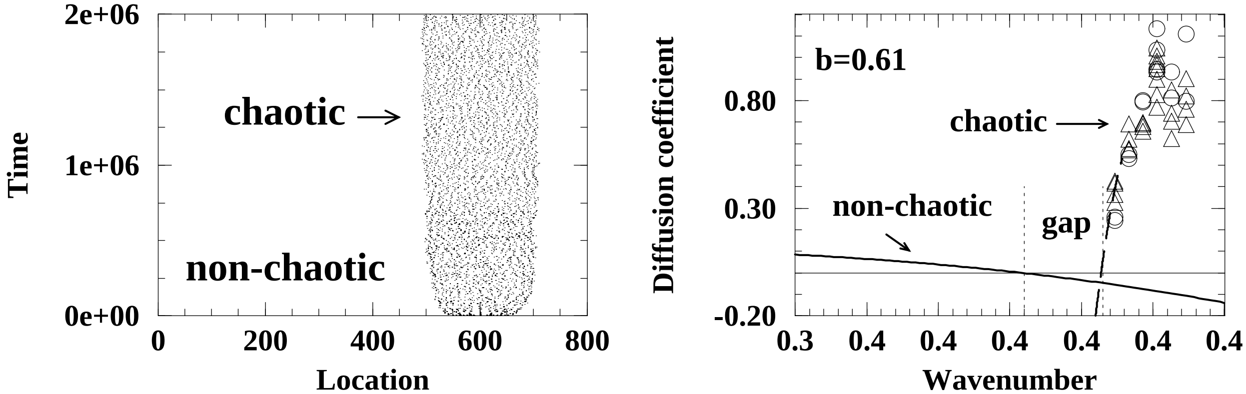


FIG. 5. Localized spatio-temporal chaos for $b = 0.6$ (other parameters as in fig.1a); the averaged effective wavenumber is $< q >= 0.377$).

FIG. 6. Diffusion coefficient of the non-chaotic and the chaotic pattern for $b = 0.6$ (other parameters as in fig.1a).

The mechanism for the localization is due to the fact that double phase slips conserve the total phase, i.e. the wavelength of the pattern is the same before and after the double phase slip. Thus, a time-averaged pattern has a well-defined wavenumber $\hat{q}$ and a well-defined phase $\hat{\phi}$. Since the phase is conserved it is expected to satisfy a slow evolution equation, which on symmetry grounds has the form of a diffusion equation

$$\partial_T \hat{\phi} = \hat{D}(\hat{q}) \partial_X^2 \hat{\phi}. \tag{5}$$

Through a detailed numerical study of the response of the extended chaotic state to a time-periodic perturbation we have explicitly demonstrated the diffusive behavior of the chaotic state on large scales [15,16]. This allowed us also to determine the effective diffusion coefficient $\hat{D}(\hat{q})$ as a function of $\hat{q}$. The result is shown in fig.6. The solid line shows the analytic result for the usual diffusion coefficient $D(q)$ (cf. (4)) which becomes negative at the Eckhaus stability limit. For larger wavenumbers the pattern is unstable to phase slips and undergoes persistent double phase slips. However, a state in which this chaotic activity is homogeneously distributed over the system is not stable to long-wave perturbations as long as the effective diffusion coefficient $\hat{D}$ is negative. Thus, there is a stability gap in wavenumber for which neither the regular nor the chaotic state are diffusively stable (see fig.6). If the initial wavenumber is chosen in that range the pattern has to break up into domains with large wavenumber, which are chaotic, and domains with smaller wavenumber, which are not chaotic. It cannot go into the chaotic (or the non-chaotic) everywhere in space since the total phase, i.e. the integral over the wavenumber, is conserved in the process. Thus, if the wavenumber increases in some part it has to decrease in another part of the system. This separation into domains is very much the same as the phase separation found in equilibrium (fluid) mixtures when they are quenched into the miscibility gap.

V. DYNAMICS IN TWO DIMENSIONS

The wavelength-changing process that occurs in one dimension *via* a phase slip involves in two dimensional patterns the creation (or annihilation) of a pair of defects (dislocations). It is therefore tempting to speculate that a double phase slip will correspond to the creation and annihilation of a 'bound defect pair', i.e. two defects will be created together and will annihilate each other soon thereafter. This would be in contrast to the dynamics observed usually (e.g. in the single complex Ginzburg-Landau equation) where the defects that are created together are not strongly correlated [17,18]. If such a regime of fluctuating bound defect pairs exists one may expect also a transition in which the defects become unbound. In thermodynamic

equilibrium such unbinding transitions have found great interest in the context of two-dimensional melting [19] and of vortex unbinding in thin-film superconductors [20]. We are currently investigating the possibility of such a transition in this non-equilibrium system.

We have obtained preliminary results that indicate that such a transition may exist for parameters corresponding to the stability diagram shown in fig.1b. Since no double phase slips seem to occur in the single Ginzburg-Landau equation describing traveling waves we consider a regime in which there exists a transition from parametrically forced standing waves to traveling waves, i.e. a parity-breaking instability. This is the case for $a_r > 0$ as shown in fig.1b. For large b the standing waves are unstable at all wavenumbers as in the case discussed above and double phase slips occur. With decreasing b the parity-breaking instability is approached and we expect that the double phase slips may become replaced by single phase slips in its vicinity.

Fig.7a,b show space-time diagrams for the y-location of defects for simulations for $b = 2$ and $a_r = -0.05$, and $b = 0.5$ and $a_r = 0.25$. For these simulations (2,3) have been extended to two dimensions by replacing the second derivative by a Laplacian. For large b (fig.7a) the defects behave as expected: after their creation they move apart, turn around and annihilate each other again, forming a loop in the space-time diagram. Even in this regime there are loops containing more than one defect pair. Fig.8 shows the statistics of loops of different sizes (solid symbols): although larger loops occur, their relative frequency decays very rapidly (exponentially) with size. For smaller b the space-time diagram of the defect location is considerably more complicated and offers no simple picture of the dynamics. The corresponding statistics of loop sizes is shown in fig.8 with open symbols. Still most loops contain only one defect pair; this reflects essentially the fact that even if the two defects that are created together moved in a completely uncorrelated fashion they would still be annihilated most likely by their 'partner' since it is closest initially. However, fig.8 shows that large loops are now considerably more frequent. In fact, a power-law decay is more consistent with the data than an exponential decay (cf. fig.8b). It should be mentioned, that the two-dimensional correlation functions of the patterns themselves show also a drastic difference: while for $b = 2$ the pattern is strongly correlated and consists of quite ordered stripes, the correlation function decays rapidly and almost isotropically for $b = 0.5$ [21]. Thus, the two regimes could also be called ordered and disordered defect chaos.

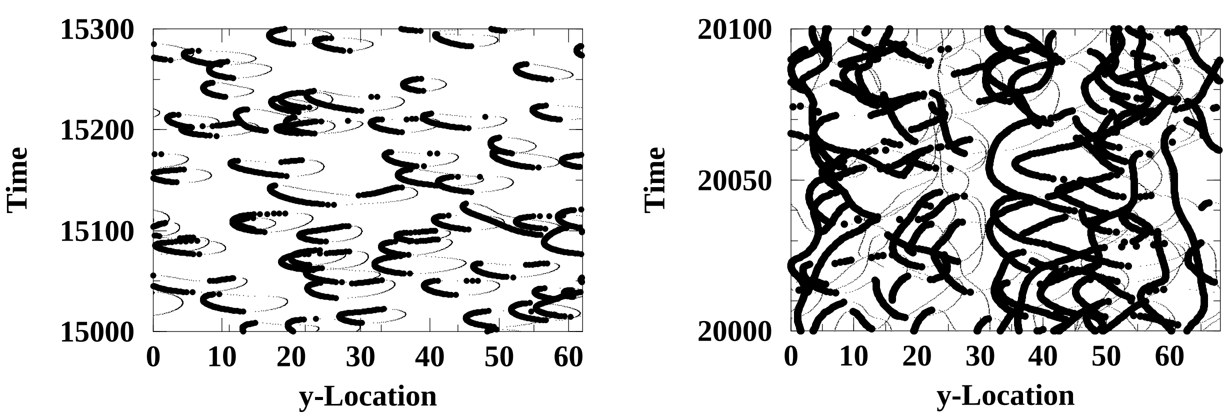


FIG. 7. y-component of the trajectory of defects. a) $b = 2$ (other parameters as in fig.1a); $b = 0.5$ (other parameters as in fig.1b).

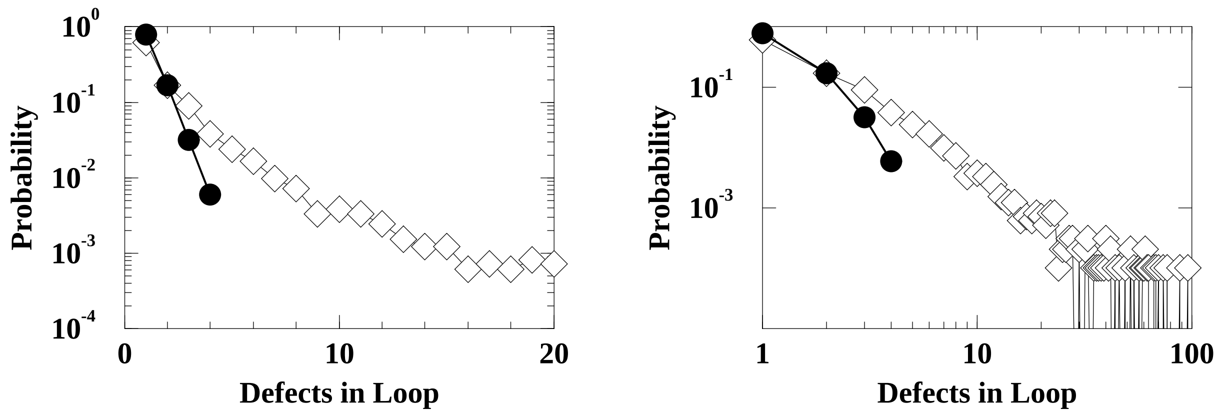


FIG. 8. Relative frequency of defect loops as a function of the number of defect pairs being part of them. open symbols: $b = 2$ (other parameters as in fig.1a); solid symbols: $b = 0.5$ (other parameters as in fig.1b). The double-logarithmic plot indicates that for $b = 0.5$ the distribution is better approximated by a power law than by an exponential.

Current and future work is directed to identify whether the change between the two regimes shown in this paper is smooth or whether it involves a true transition. As diagnostics we are not only using the loop-size distribution, but also the defect life-time, the spatial extent of the loops, the distances traveled by each defect, as well as correlation functions.

This work was supported by the United States Department of Energy through grant DE-FG02-92ER14303. It also made use of the resources of the Cornell Theory Center, which receives major funding from NSF and New York State with additional support from the Advanced Research Projects Agency, the National Center for Research Resources at the National Institutes of Health, IBM Corporation and members of the Corporate Research Institute.

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